

On graphs related to co-maximal ideals of a commutative ring^{*}

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Abstract. This paper studies the co-maximal graph $\Omega(R)$, the induced subgraph $\Gamma(R)$ of $\Omega(R)$ whose vertex set is $R \setminus (U(R) \cup J(R))$ and a retract $\Gamma_r(R)$ of $\Gamma(R)$, where R is a commutative ring. We show that the core of $\Gamma(R)$ is a union of triangles and rectangles, while a vertex in $\Gamma(R)$ is either an end vertex or a vertex in the core. For a non-local ring R , we prove that both the chromatic number and clique number of $\Gamma(R)$ are identical with the number of maximal ideals of R . A graph $\Gamma_r(R)$ is also introduced on the vertex set $\{Rx \mid x \in R \setminus (U(R) \cup J(R))\}$, and graph properties of $\Gamma_r(R)$ are studied.

Key Words: Co-maximal graph; Split graph; Core; Chromatic number; Retract of a graph.

1. Introduction

In 1988, Beck [4] introduced the concept of zero-divisor graph for a commutative ring. Since then a lot of work was done in this area of research. Several other graph structures were also defined on rings and semigroups. In 1995, Sharma and Bhatwadekar [10] introduced a graph $\Omega(R)$ on a commutative ring R , whose vertices are elements of R where

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two distinct vertices x and y are adjacent if and only if $Rx + Ry = R$. Recently, Maimani et.al. in [8] named this graph $\Omega(R)$ as the *co-maximal graph* of R and they noticed that the subgraph $\Gamma(R)$ induced on the subset $R \setminus (U(R) \cup J(R))$ is the key to the co-maximal graph. Many interesting results about the subgraph were obtained in [8] and Wang [11], and their work show that the properties of the graph $\Gamma(R)$ are quite similar to that of the modified zero-divisor graph by Anderson and Livingston [2]. For example, both graphs are simple, connected and with diameter less than or equal to three, and each has girth less than or equal to four if they contain a cycle. Because of this reason, in this paper we use $\Gamma(R)$ to denote the graph $\Gamma_2(R) \setminus J(R)$ of [8]. We discover more properties shared by both zero-divisor graph and the subgraph $\Gamma(R)$ of $\Omega(R)$. In particular, It is shown that the core of $\Gamma(R)$ is a union of triangles and rectangles, while a vertex in $\Gamma(R)$ is either an end vertex or a vertex in the core. For any non-local ring R , it is shown that the chromatic number of the graph $\Gamma(R)$ is identical with the number of maximal ideals of R . In Section 4, we introduce a new graph $\Gamma_r(R)$ on the vertex set

$$\{Rx \mid x \in R \setminus (U(R) \cup J(R))\}.$$

This graph is in fact a retract of the graph $\Gamma(R)$ and thus simpler than the graph $\Gamma(R)$ in general, but we will show that they share many common properties and invariants.

Jinnah and Mathew in [6] studied the problem of when a co-maximal graph $\Omega(R)$ is a split graph, and they determined all rings R with the property. In Section 2, we give an alternative proof to their Theorem 2.3. In the co-maximal graph $\Omega(R)$, each unit u of R is adjacent to all vertices of the graph while an element of $J(R)$ only connects to units of R . Temporally, we say u is in the center of the graph $\Gamma(R)$. Related to the co-maximal relation, there is the concept of *rings with stable range one*. Recall that a ring R (which needs not be commutative) has one in its stable range, if for any x, y with $Rx + Ry = R$, there is an element t such that $x + ty$ is invertible. For example, the following classes of rings have one in their stable range: zero-dimensional commutative rings, von Neumann unit-regular rings, semilocal rings. The concept co-maximal graph gives an interesting graph interpretation of such rings. In fact, *a commutative ring R has one in its stable range if and only if for any pair of adjacent vertices x, y in the co-maximal graph $\Omega(R)$, the additive coset $x + Ry$ (and $y + Rx$) has at least one element in the center of the graph $\Omega(R)$.*

Throughout this paper, all rings are assumed to be commutative with identity. For a ring R , let $U(R)$ be the set of invertible elements of R and $J(R)$ the Jacobson radical of R . Recall that a graph is called *complete* (*discrete*, respectively) if every pair of vertices are adjacent (respectively, no pair of vertices are adjacent). We denote a complete graph by K , a complete (discrete, resp.) graph with n vertices by K_n (resp., D_n). A subset K of the vertex set of G is called a *clique* if any two distinct vertices of K are adjacent; the *clique number* $\omega(G)$ of G is the least upper bound of the size of the cliques. Similarly, we denote by $K_{m,n}$ the complete bipartite graph with two partitions of sizes m, n respectively.

Recall that a simple graph G is called a *refinement* of a simple graph H if $V(G) = V(H)$ and $a - b$ in H implies $a - b$ in G for all distinct vertices of G , where $a - b$ means that $a \neq b$ and a is adjacent to b . Recall that a *cycle* in a graph is a path $v_1 - v_2 - \cdots - v_n$ together with an additional edge $v_n - v_1$ ($n \geq 3$). For a simple graph G and a nonempty subset S of $V(G)$, there is *the subgraph induced on S* : the vertex set is S and the edge set is

$$\{x - y \mid x \neq y \in S, \text{ and there is an edge } x - y \text{ in the graph } G\}.$$

A discrete induced subgraph of a graph G is also called an *independent subset* of G .

2. Rings R whose co-maximal graph is a split graph

Throughout this section, assume that G is a *split graph*, i.e., G is simple and connected with $V(G) = K \cup D$, where $K \cap D = \emptyset$ and the induced subgraph on K (respectively, on D) is a complete (discrete, respectively) graph. For the split graph G , we always assume that D is a maximal such independent subset. Under the assumption, a complete graph K_n is a split graph with $|K| = n - 1$, $|D| = 1$.

Lemma 2.1. *For a commutative ring R , let G be the co-maximal graph $\Omega(R)$. If G is a split graph with $V(G) = K \cup D$, then*

- (1) *For any proper ideal I of R , $|I \cap K| \leq 1$.*
- (2) *$\mathfrak{m} \cap \mathfrak{n} \cap K = \{0\}$ holds for distinct maximal ideals $\mathfrak{m}, \mathfrak{n}$ of R .*
- (3) *If R is isomorphic to neither \mathbb{Z}_2 nor $\mathbb{Z}_2 \times \mathbb{Z}_2$, then $|K| \geq |\text{Max}(R)| + 1$. Also, $|K| = |\text{Max}(R)|$ iff $R = \mathbb{Z}_2$ or $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.*

Proof. (1) Clear.

(2). Assume to the contrary that there exists some nonzero $u \in \mathfrak{m} \cap \mathfrak{n} \cap K$. Since $\mathfrak{m} + \mathfrak{n} = R$, we have $x \in \mathfrak{m}, y \in \mathfrak{n}$ such that $x + y = 1$. Clearly, $x \neq y$ and hence we can assume $x \in K$. Then it follows by (1) that $x = u \in \mathfrak{n}$, a contradiction.

(3) If R is a local ring, then either $R \cong \mathbb{Z}_2$ or R has at least two units by assumption. The result holds in this case. In the following, we assume that R is non-local. If there is a maximal ideal \mathfrak{m} with $\mathfrak{m} \subseteq D$, then by the proof of [6, Theorem 2.1], $R \cong \mathbb{Z}_2 \times \mathbb{F}$ for some field \mathbb{F} . In this case, $|K| \geq |\text{Max}(R)|$ and, $|K| = |\text{Max}(R)|$ iff $\mathbb{F} = \mathbb{Z}_2$. If no maximal ideal is contained in K , then each maximal ideal has exactly one vertex in K by (1). By (2), we have $|K| \geq |\text{Max}(R)| + 1$. ■

We remark that Lemma 2.1 (3) is the best possible result. For instance, $|K| = |\text{Max}(R)| + 1$ holds for the local ring \mathbb{Z}_4 and $\prod_1^3 \mathbb{Z}_2$. For $R = \prod_1^3 \mathbb{Z}_2$, we draw its co-maximal graph $\Omega(R)$ in Figure 1, in which $\Omega(R) = K_1 + K_1 + H$.

The following is a result of [6]:

Lemma 2.2. ([6, Theorem 2.1]) *For a commutative ring R which is non-local, if the co-*

maximal graph $\Omega(R)$ is a split graph and $R \not\cong \mathbb{Z}_2 \times \mathbb{F}$ for any field \mathbb{F} , then the characteristic of R is two, R has exactly three maximal ideal \mathfrak{m}_i , and $K \cap \mathfrak{m}_i = \{x_i\}$ for all i , where each x_i is idempotent and for every invertible element u of R , $ux_i = x_i$.

We now prove give an alternative proof to the main result of [6]:

Theorem 2.3. ([6, Theorem 2.3]) *For a commutative ring R , the co-maximal graph $\Omega(R)$ is a split graph if and only if R is one of the following: a local ring, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{F}$ for some field \mathbb{F} .*

Proof. We only need prove the necessary part. Assume that R is not a local ring and $R \not\cong \mathbb{Z}_2 \times \mathbb{F}$ for any field \mathbb{F} . Let $G = \Omega(R)$. Assume further that the co-maximal graph G is a split graph with $V(G) = K \cup D$. By Lemma 2.2, R has exactly three maximal ideals \mathfrak{m}_i ($1 \leq i \leq 3$) such that $\mathfrak{m}_i \cap K = \{x_i\}$, where $x_i^2 = x_i$ and $ux_i = x_i$ holds for every $u \in U(R)$. Also, R has characteristic 2. For any $1 \leq i \neq j \leq 3$, assume $1 = rx_i + sx_j$ with $rx_i \in K$. By Lemma 2.1 we have $x_i = rx_i$ and hence $1 - x_i = sx_j$. It follows that $(1 - x_i)(1 - x_j) = 0$. Notice that $1 + x_1x_2x_3 \in U(R)$ since $x_1x_2x_3 \in J(R)$, it follows that $x_1x_2x_3 = 0$. Then $(1 - x_i)(1 - x_j) = 0$ implies $x_k + x_ix_k + x_jx_k = 0$ for any rearrangement i, j, k of $1, 2, 3$. Therefore $x_1 + x_2 + x_3 = 2 \cdot f(x_1, x_2, x_3) = 0$. This shows that $1 - x_1, 1 - x_2, 1 - x_3$ is a complete set of orthogonal idempotent elements of R .

Let $R_i = R(1 - x_i)$. Then $R = \sum_{i=1}^3 R_i \cong R_1 \times R_2 \times R_3$, where each R_i is a local ring with a unique maximal ideal \mathfrak{n}_i , since R has exactly three maximal ideals. For any $r(1 - x_1) \in \mathfrak{n}_1$, we have

$$1 + r(1 - x_1) = [(1 - x_1) + r(1 - x_1)] + (1 - x_2) + (1 - x_3) \in U(R).$$

Since $ux_2 = x_2$ for all $u \in U(R)$, it follows that $r(1 - x_1) \cdot x_2 = 0$. Now apply the fact $(1 - x_1)(1 - x_2) = 0$, one derives $r(1 - x_1) = 0$. Hence $\mathfrak{n}_1 = 0$ and each R_i is a field. Thus R is a direct product of three fields.

Since $x_1 = x_2 + x_3$, it follows that

$$\mathfrak{m}_1 = R_2 + R_3, \mathfrak{m}_2 = R_1 + R_3, \mathfrak{m}_3 = R_1 + R_2.$$

Then in the decomposition $R = K \cup D$, we deduce from Lemma 2.1 (1) that

$$K = (U(R_1) + U(R_2) + U(R_3)) \cup \{x_1, x_2, x_3\}, D = R \setminus K.$$

Now we claim that each R_i is isomorphic to \mathbb{Z}_2 and hence, $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. In fact, assume $R = R_1 \times R_2 \times R_3$. Then

$$K = (U(R_1) \times U(R_2) \times U(R_3)) \cup \{(0, 1, 1), (1, 0, 1), (1, 1, 0)\}.$$

If $|R_1| > 2$, then there exists a nonzero element $1 \neq v_1 \in R_1$. Then both $z = (v_1, 0, 1)$ and $e = (0, 1, 0)$ are in the independent subset D , contradicting $Rz + Re = R$. This completes

the proof. ■

Recall a convenient construction from graph theory, *the sequential sum*

$$G_1 + G_2 + \cdots + G_r$$

of a sequence of graphs G_1, G_2, \dots, G_r . We illustrate the construction in Figure 1 for the sequence of graphs K_1, K_1, H , where H is a triangle K_3 together with three end vertices adjacent to distinct vertices.

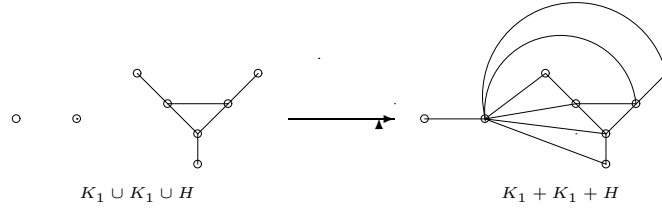


Figure 1. Sequential sum

Corollary 2.4. *A finite split graph is a co-maximal graph of a non-local commutative ring if and only if G is one of the following: (1) The sequential sum $K_1 + K_{p^n-1} + K_{1,p^n-1}$ for some prime number p . (2) The sequential sum $K_1 + K_1 + H$ in Figure 1, where H is a triangle K_3 together with three end vertices adjacent to distinct vertices.*

The co-maximal graph of a field is certainly a complete graph. For a finite local ring (R, \mathfrak{m}) which is not a field, assume $|\mathfrak{m}| = p^m$. Then the co-maximal graph of R is $K_{p^n-p^m} + D_{p^m}$, where $p^n \geq 2p^m$ and D_{p^m} is a discrete graph with p^m vertices.

3. The subgraph $\Gamma(R)$

As noticed by Maimani et.al. in [8], the main part of the co-maximal graph $\Omega(R)$ is the subgraph $\Gamma(R)$ induced on the vertex subset $R \setminus (U(R) \cup J(R))$. In fact, we have the following facts:

(O1) A vertex in $U(R)$ is adjacent to every vertex of $\Omega(R)$, while an element of $J(R)$ only connects to units of R . In fact, there is a sequential sum decomposition

$$\Omega(R) = J(R) + U(R) + \Gamma(R).$$

(O2) $\Gamma(R)$ is empty if and only if R is a local ring, i.e., a commutative ring with a unique maximal ideal.

[8, 11] studied this subgraph and obtained many interesting results. In particular,

it is proved that the graph is connected with diameter less than or equal to three ([8, Theorem 3.1]), that the girth of the graph is less than or equal to four ([11, Corollary 3.8]). We include a detailed proof for the following fundamental property of graphs related to algebras:

Theorem 3.1. ([8, Theorem 3.1]) *The graph $\Gamma(R)$ is connected with diameter less than or equal to three.*

Proof. For any $a \in R$, set $S_a = \{\mathfrak{m} \in \text{Max}(R) \mid a \in \mathfrak{m}\}$. Then for each $a \notin J(R)$, $S_a \subset \text{Max}(R)$. For distinct $a, b \in R$, we claim

- (1) $ab \in J(R)$ iff $\text{Max}(R) = S_a \cup S_b$,
- (2) $Ra + Rb = R$ iff $S_a \cap S_b = \emptyset$.

Now for distinct $a, b \in R \setminus (U(R) \cup J(R))$, if $ab \notin J(R)$, then there exists $x \in R \setminus (U(R) \cup J(R))$ such that $Rab + Rx = R$. Then clearly there is a path $a - x - b$ in $\Gamma(R)$ and hence $d(a, b) \leq 2$. If $ab \in J(R)$, then take any $y \in R \setminus (U(R) \cup J(R))$ such that $Ra + Ry = R$. We claim that $by \notin J(R)$ and it will follow that $d(a, b) \leq d(a, y) + d(y, b) \leq 3$. In fact, assume to the contrary that $by \in J(R)$. Then we have

$$S_b \cup (S_y \setminus S_b) = S_b \cup S_y = \text{Max}(R) = S_b \cup (S_a \setminus S_b).$$

It follows that $(S_y \setminus S_b) = S_a \setminus S_b = \emptyset$ since $S_a \cap S_y = \emptyset$. Then $b \in J(R)$, a contradiction. This completes the proof. ■

By [11, Theorem 3.9(1)], the clique number $\omega(\Gamma(R))$ is infinite whenever $J(R) = 0$ and the ring R is indecomposable. It could be used to sharpen [8, Theorem 2.2], as the following theorem shows.

Theorem 3.2. *For any non-local ring R , let $G = \Gamma(R)$. Then the following are equivalent:*

- (1) G is a bipartite graph.
- (2) G is a complete bipartite graph.
- (3) R has exactly two maximal ideals.
- (4) $R/J(R) \cong \mathbb{K}_1 \times \mathbb{K}_2$, where each \mathbb{K}_i is a field.

Proof. (3) \implies (2): Assume that $\mathfrak{m}_1, \mathfrak{m}_2$ are the maximal ideals of R . Then $\Gamma(R)$ is a complete bipartite graph with two partitions $\mathfrak{m}_1 \setminus \mathfrak{m}_2$ and $\mathfrak{m}_2 \setminus \mathfrak{m}_1$.

(2) \implies (3): Assume that G is a complete bipartite graph with vertex partition $V(G) = V_1 \cup V_2$. Then for any maximal ideal \mathfrak{m} , $\mathfrak{m} \setminus J(R)$ is entirely contained in a single partition. Assume $\mathfrak{m}_i \setminus J(R) \subseteq V_i$. If R has a third maximal ideal \mathfrak{n} , then $\mathfrak{n} \setminus J(R) \subseteq V_i$ for some i . This is impossible since $\mathfrak{n} + \mathfrak{m}_i = R$.

(2) \implies (1) and (4) \implies (3): Clear.

(1) \implies (3): Clearly, there is no loss to assume $J(R) = 0$.

If R has only a finite number of maximal ideals, say \mathfrak{m}_i , $1 \leq i \leq r$, set $\mathfrak{n}_i = \mathfrak{m}_i \setminus (\cup_{j \neq i} \mathfrak{m}_j)$. Then $\mathfrak{n}_i \neq \emptyset$, and each vertex in \mathfrak{n}_i is adjacent to all vertex in \mathfrak{n}_j . This implies $\omega(G) \geq |Max(R)|$ and hence $\omega(G) = |Max(R)|$, when R has only a finite number of maximal ideals. If further G is a bipartite graph, then $r = 2$, i.e., R has exactly two maximal ideals.

In the following, assume that R has infinitely many maximal ideals, and we proceed to prove $\omega(G) = \infty$. Assume that R has a non-trivial idempotent e . Then $R = eR \times (1-e)R$. If eR has no nontrivial idempotent element, then $\omega(\Gamma(eR)) = \infty$ and hence $\omega(G) = \infty$. Then assume that both e and $1-e$ are non-primitive idempotents. By induction, for any integer $s \geq 1$, there exist non-trivial corner rings $R_{s,i}$ of R such that $R = \prod_{j=1}^s R_{s,j}$. Set

$$f_1 = (0, 1, \dots, 1), \dots, f_2 = (1, 0, 1, \dots, 1), \dots, f_s = (1, \dots, 1, 0).$$

Clearly $\{f_j\}_{1 \leq j \leq s}$ is a clique in $\Gamma(R)$, and thus $\omega(G) = \infty$.

(3) \implies (4): This follows from the Chinese Remainder Theorem. ■

Notice that the proof together with [11, Theorem 3.9(1)] actually gives an alternative proof to the fact that $\omega(\Gamma(R)) = |Max(R)|$ whenever R is not a local ring.

Recall that a ring R is called an *exchange ring* if the left module ${}_R R$ has the exchange property, see [12] and the included references for details. Recall that idempotents can be lifted modulo every ideal of an exchange ring R . The class of exchange rings include artinian rings, semiperfect rings and clean rings, the rings in which each element is a sum of an idempotent and a unit. Recall that for a ring R with all idempotents central in R , R is clean iff R is an exchange ring. For commutative clean rings R , we have

Corollary 3.3. *For any commutative non-local exchange ring R , let $G = \Gamma(R)$. Then G is a bipartite graph iff G is a complete bipartite graph, if and only if $R \cong R_1 \times R_2$, where each R_i is a local ring.*

A graph G is called *totally disconnected* if the edge set $E(G)$ is empty. By Theorem 3.1, we have the following observation and hence Theorem 3.4:

(O3) $\Gamma(R)$ is totally disconnected iff it is an empty graph, if and only if R is a local ring.

Theorem 3.4. ([8, 11]) *For a ring R , let $G = \Gamma(R)$. Then the following are equivalent:*

- (1) G is a refinement of a star graph, i.e., G has at least two vertices, and there exists a vertex in G which is adjacent to every other vertex.
- (2) G is a tree, i.e., G is nonempty, connected and contains no cycles.
- (3) G is a star graph.
- (4) R is isomorphic to $\mathbb{Z}_2 \times \mathbb{F}$ for some field \mathbb{F} .

Proof. (1) \iff (3) \iff (4): This is contained in [8, Corollary 2.4(2)].

(2) \iff (4): This is contained in [11, Theorem 3.5, Corollary 3.6]. ■

Corollary 3.5. *For a finite simple graph G with $|G| \geq 2$, assume $G = \Gamma(R)$ for some ring R . Then the following are equivalent:*

- (1) G is a refinement of a star graph.
- (2) G is a tree, i.e., G contains no cycle.
- (5) $G = K_{1,p^n-1}$ for some prime number p and some positive integer n .

By the result of section 2 and the results of [8, 11], it is natural to ask the following question: For what rings R is $\Gamma(R)$ a split graph? Notice that for distinct maximal ideals $\mathfrak{m}, \mathfrak{n}$ (if exist) and $x \in \mathfrak{m}, y \in \mathfrak{n}$, $Rx + Ry = R$ implies $x, y \in R \setminus (U(R) \cup J(R))$. Then a careful check to the proofs of Lemma 2.1, Lemma 2.2 and Theorem 2.3 shows the following:

Theorem 3.6. *For any ring R , the following statements are equivalent:*

- (1) *The co-maximal graph $\Omega(R)$ of R is a split graph.*
- (2) *The subgraph $\Gamma(R)$ is either empty or a split graph.*
- (3) *R is one of the following: a local ring, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{F}$ for some field \mathbb{F} .*

Notice that a split graph G is isomorphic to $\Gamma(R)$ for some finite ring R iff either G is a star graph K_{1,p^n-1} for some prime number p , or G is the triangle together with three end vertices adjacent to distinct vertices (see Figure 1).

The works of [8, 11] show that the graph $\Gamma(R)$ has many properties which the zero-divisor graph of a ring (or a semigroup) already have. Recall from [5, 9] that *the core of a zero-divisor graph G is always a union of triangles and squares, and a vertex in G is either an end vertex or a vertex of the core*. Recall that the core of a graph G is by definition the subgraph induced on all vertices of cycles of G . In the final part of this section, we will show that the graph $\Gamma(R)$ has the same property.

Lemma 3.7. *For any path $a - x - b$ in the graph $\Gamma(R)$, if $ab + x$ is not a unit of R , then the path is contained in a subgraph isomorphic to $K_1 + K_2 + K_1$.*

Proof. The given condition implies $Rx + Rab = R$. Hence $ab \notin J(R)$ and in particular, $ab + x \neq x$. Furthermore, we have

$$R(ab + x) + Ra = R = R(ab + x) + Rb.$$

Now assume $ab + x \notin U(R)$. Then $ab + x \notin U(R) \cup J(R)$. If $ab + x \notin \{a, b\}$, then there is a subgraph $K_1 + K_2 + K_1$ in $\Gamma(R)$ which contains the path $a - x - b$, see Figure 2. Since $R(ab + x) + Ra = R = R(ab + x) + Rb$, it follows that $ab + x \notin \{a, b\}$, and this completes the proof. ■

Lemma 3.8. *If a vertex x of $\Gamma(R)$ is in a cycle of five vertices, then x is in either a*

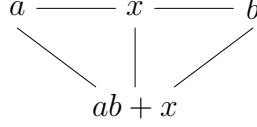


Figure 2. Lemma 3.7

triangle or a rectangle.

Proof. Assume $a - x - b - c - d - a$ is a cycle in $\Gamma(R)$, and x is not in any triangle. We proceed to verify that x is in a square.

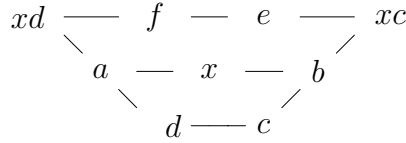


Figure 3. Lemma 3.8

In fact, if $xv \neq x$ for some $v \in U(R)$, then there is a square $a - x - b - vx - a$ in $\Gamma(R)$. Thus in the following, we assume $uy = y$, $\forall u \in U(R)$ where $y \in \{a, x, b\}$. Then by Lemma 3.7, we can assume further

$$ab + x = 1, xc + b = 1, xd + a = 1 \quad (*).$$

Notice that xd is adjacent to a , and xc is adjacent to b in $\Gamma(R)$. If $xd = x$, then by $(*)$ we have $a = ab$. Then there is a rectangle $a - x - b - d - a$ in $\Gamma(R)$. Therefore assume $xd \notin \{x, d\}$, $xc \notin \{x, c\}$.

Since $d(xc, xd) \leq 3$, we can assume there is a path $xc - e - f - xd$. If $e \neq b$, then there is a rectangle $x - e - xc - b - x$. If $e = b$, then $f \notin \{x, a\}$ and hence by $(*)$ there is a cycle $a - x - b - f - xd - a$. Then there is a rectangle $a - x - f - xd - a$ and a triangle $b - x - f - b$. This completes the verification. ■

Lemma 3.9. *Let $G = \Gamma(R)$ and assume that G contains a cycle. Then for any path $a - x - b$ in the core of G , x is in a cycle C_n with $3 \leq n \leq 5$.*

Proof. Assume

$$\dots - f - a - x - b - g - \dots$$

is a cycle in $\Gamma(R)$. We proceed to verify that x is in a cycle C_n with $3 \leq n \leq 5$. Since $Rab + Rfb = Rb$, we have

$$Rb = Rab + (Ra + Rx)fb = Rab + Rxfb.$$

Then $Rb + R(x - xfb) = Rab + Rx = R$. It follows that $x - xfb \notin U(R) \cup J(R)$. Consider xfb . If $xfb = 0$, then it follows that $Rb = Rab + Rxfb = Rab$. In this case, there is a rectangle $a - x - b - g - a$. In the following, assume $xfb \neq 0$, i.e., $x - xfb \neq x$, and let $y = x - xfb$. Without loss of generality, assume $d(y, f) = 3$ and $y - d - e - f$ is a path from y to f . If further $d \neq b$, then there is a rectangle $b - x - d - y - b$. If $d = b$, then there is a cycle $C_5 : f - a - x - b - e - f$. This completes the proof ■

Theorem 3.10. For a ring R , let $G = \Gamma(R)$ and assume that G contains a cycle. Then the core of G is a union of triangles and rectangles, and every vertex of G is either an end vertex or a vertex of the core.

Proof. The first statement follows from Lemmas 3.9 and 3.8. The second statement follows after an argument similar to the proof of [5, Theorem 1.5]. We omit the details here. ■

4. A retract $\Gamma_r(R)$ of $\Gamma(R)$

For simple graphs G and H , recall that $G \xrightarrow{\varphi} H$ if there exists a map $\varphi : G \rightarrow H$ such that for distinct $u, v \in V(G)$, $u - v$ in G implies $\varphi(u) \neq \varphi(v)$ and $\varphi(u) - \varphi(v)$ in H . Such a map is called a *graph homomorphism*. If H is a subgraph of G , $G \xrightarrow{\varphi} H$ and the restriction of φ on H is an identity, then H is called a *retract* of G , see Figure 4 for an example. If G has no proper retract, then G is called a *core graph* (e.g., K_n is a core graph.). Notice that the core of a graph needs not be a core graph. Recall that the *chromatic number* $\chi(G)$ of G is the least positive integer r such that $G \rightarrow K_r$. It is the least number of colors needed for coloring the vertices of G in such a way that no two adjacent vertices have a same color. The girth of a graph G , denoted by $g(G)$, is the length of the minimal cycle in G .

In this section, we introduce a new graph which is a retract of $\Gamma(R)$ and we study this new graph.

Definition 4.1. For a ring R and any $x \in R$, let $\bar{x} = Rx$. Construct a simple graph in the following and denote it as $\Gamma_r(R) = G$:

$$V(G) = \{\bar{x} \mid x \in R \setminus (U(R) \cup J(R))\},$$

$$E(G) = \{\{\bar{x}, \bar{y}\} \mid \bar{x} \neq \bar{y} \in V(G), \text{ and } Rx + Ry = R\}$$

Clearly, this graph has less vertices and less edges than that of the graph $\Gamma(R)$ in general, see Figure 4. More precisely,

Proposition 4.2. (1) For a ring R , the graph $\Gamma_r(R)$ is a retract of $\Gamma(R)$.
(2) For any ideal I of R contained in $J(R)$, $\Gamma(R/I)$ is a retract of $\Gamma(R)$.
Furthermore,

- (3) If the girth of $\Gamma(R)$ is three, then so is the girth of $\Gamma_r(R)$.
- (4) $\Gamma_r(R)$ is connected with diameter less than or equal to three.
- (5) $\omega(\Gamma_r(R)) = \omega(\Gamma(R))$ and $\chi(\Gamma_r(R)) = \chi(\Gamma(R))$.
- (6) $\Gamma_r(R)$ and $\Gamma(R)$ are homomorphically equivalent to a same unique core graph.

Proof. (1) Clearly, $\varphi : x \mapsto \bar{x}$ is a surjective graph homomorphism. For any $\bar{x} \in V(\Gamma_r(R))$, fix a vertex y_x in Rx to obtain a subgraph of $\Gamma(R)$. Then the graph $\Gamma_r(R)$ is isomorphic to the subgraph of $\Gamma(R)$. Thus $\Gamma_r(R)$ is a retract of the graph $\Gamma(R)$. In a similar way, one checks (2). All the remaining results follow from (1). We check (3) and (5) in the following.

(3) The result is clear by the definition of $\Gamma_r(R)$. In fact, if $G \rightarrow H$ and G has odd girth, then it is known that $g(G) = g(H)$, since any cycle with odd length is a core graph.

(5) Since a composition of graph homomorphisms is still a graph homomorphism, clearly $\chi(\Gamma_r(R)) \geq \chi(\Gamma(R))$. On the other hand, since $\Gamma_r(R)$ is isomorphic to a subgraph of $\Gamma(R)$, for any graph homomorphism $\psi : \Gamma(R) \rightarrow K_s$, the restriction $\psi| : \Gamma_r(R) \rightarrow K_s$ is also a graph homomorphism. This shows the second assertion of (5). The first equality holds by the definition of a retract of a graph. ■

By Theorem 3.2 and Proposition 4.2, we have

Corollary 4.3. (1) For any non-local ring R , $\chi(\Gamma_r(R)) \geq |Max(R)|$.

(2) $\Gamma_r(R)$ is a complete bipartite graph iff R has exactly two maximal ideals, iff $\Gamma_r(R)$ is a bipartite graph.

By [11, Corollary 3.8(1)], if $H = \Gamma(R)$ contains a cycle, then the girth $g(H) \leq 4$. Compared with Proposition 4.2(3), we have the following example for the $g(H) = 4$ case:

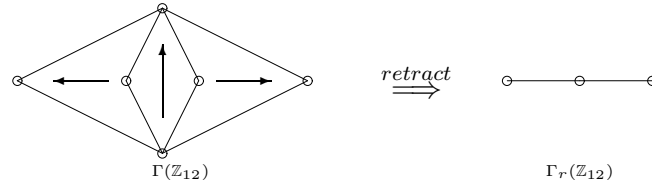


Figure 4. Retract for ring \mathbb{Z}_{12}

Example 4.4. Consider the ring $R = \mathbb{Z}_{12}$. For this ring, $g(\Gamma(R)) = 4$, while $g(\Gamma_r(R)) = \infty$. We draw the two graphs in Figure 4.

Theorem 4.5. For any commutative ring R which is not a local ring, let $G = \Gamma(R)$. Then the following numbers are identical:

- (1) The chromatic number $\chi(G)$;

- (2) The clique number $\omega(G)$;
- (3) The cardinal number of $Max(R)$;
- (4) $\chi(\Gamma_r(R))$;
- (5) $\omega(\Gamma_r(R))$.

Proof. (1) First, recall from [10, Theorem 2.3] that

$$\chi(\Omega(R)) = |Max(R)| + |U(R)|.$$

Then consider the following decomposition of the co-maximal graph into sequential sums of three subgraphs

$$\Omega(R) = J(R) + U(R) + \Gamma(R),$$

where $J(R)$ is a discrete subgraph while $U(R)$ is a complete subgraph. Since $|J(R)| \leq |U(R)|$, we have

$$\chi(\Omega(R)) = \chi(\Gamma(R)) + |U(R)|.$$

By [11, Theorem 3.9(2)], $\omega(\Gamma(R)) = |Max(R)|$. Then

$$\chi(\Gamma(R)) = |Max(R)| = \omega(\Gamma(R))$$

(2) By Lemma 4.2, we have $\chi(G) = \chi(\Gamma_r(R)) \geq \omega(\Gamma_r(R)) = \omega(G)$, and the result follows from (1). ■

Now we study the interplay between the graph structure of $\Gamma_r(R)$ and the algebraic property of R . First we have

Proposition 4.6. *For a ring R , let $G = \Gamma_r(R)$. Then the following are equivalent:*

- (1) G is a refinement of a star graph.
- (2) G is a star graph.
- (3) $R \cong \mathbb{F} \times T$, where \mathbb{F} is a field and T is a local ring.

Proof. We only need prove (1) \implies (3). Assume \bar{x} is adjacent to every other vertex in $\Gamma_r(R)$. Then $\bar{x} = \bar{x}^2$. Assume $x = rx^2$. Then $rx = (rx)^2$ and $\bar{x} = \overline{rx}$. So we can assume at the start that x is a nontrivial idempotent. Let $Rx = T$, $\mathbb{F} = R(1 - x)$. Then T is a maximal ideal of R and $R = \mathbb{F} \times T$. Thus \mathbb{F} is a field. Now $J(R) = J(T)$, and for any $y \in T \setminus J(T)$, we have $T = Ty$. Thus T is a local ring with the maximal ideal $J(R)$. ■

Notice that for a finite local ring (T, \mathfrak{m}) , if $\mathfrak{m} = Tx_1 \cup \dots \cup Tx_r$ in which $Tx_i \neq Tx_j$, then $\Gamma_r(R) = K_{1,r}$.

Corollary 4.7. *For any commutative ring R , $\text{diam}(\Gamma_r(R)) = 1$ iff $R \cong \mathbb{F}_1 \times \mathbb{F}_2$, where each \mathbb{F}_i is a field.*

Proof. If $R \cong \mathbb{F}_1 \times \mathbb{F}_2$, then clearly $\text{diam}(\Gamma_r(R)) = 1$. Conversely, assume $\text{diam}(\Gamma_r(R)) = 1$. Then $\Gamma_r(R)$ is a complete graph. By Proposition 4.6, we have $R \cong \mathbb{F} \times T$, where \mathbb{F} is

a field and T is a local ring. If T is not a field, then take any nonzero $x \in J(T)$. Then $(1, x) \in R \setminus U(R) \setminus J(R)$, and $\overline{(1, 0)}$ is not adjacent to $\overline{(1, x)}$ in $\Gamma_r(R)$, a contradiction. This completes the proof. ■

The following result follows from Corollary 4.3, Proposition 4.6 and the proof of [8, Lemma 3.2, Proposition 3.3]:

Proposition 4.8. *For any commutative non-local ring R , $\text{diam}(\Gamma_r(R)) = 2$ iff one of the following conditions holds:*

- (1) $J(R)$ is a prime ideal of R .
- (2) R has exactly two maximal ideals, and $R \not\cong \mathbb{F}_1 \times \mathbb{F}_2$ for any fields \mathbb{F}_i .

Corollary 4.9. *For any commutative non-local ring R , the graphs $\Gamma(R)$ and $\Gamma_r(R)$ have a same diameter iff $R \not\cong \mathbb{F}_1 \times \mathbb{F}_2$ for any fields \mathbb{F}_i but $\mathbb{F}_1 = \mathbb{F}_2 = \mathbb{Z}_2$.*

By [11, Theorem 3.9(2)], Theorem 4.5 and Corollary 4.7, if $\text{diam}(\Gamma_r(R)) = 2$ (respectively, $\text{diam}(\Gamma(R)) = 2$), then either $\Gamma_r(R)$ (respectively, $\Gamma(R)$) is a complete bipartite graph or its clique number is infinite.

At the end of the paper, we pose the following problem:

Question 4.10. *Which rings R have the property that $\Gamma(R)$ is a generalized split graph? Which rings R have the property that $\Gamma_r(R)$ is a (generalized) split graph?*

Recall from [7] that a simple graph G is a *generalized split graph* if

$$V(G) = K \cup D, K \cap D = \emptyset,$$

where the induced subgraph on K (resp., on D) is a core graph (respectively, a discrete graph). Notice that $\Gamma(\mathbb{Z}_{12})$ is not a generalized split graph, while $\Gamma_r(\mathbb{Z}_{12})$ is a split graph.

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